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Galois extensions and maps on local cohomology

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Abstract: We improve the Main theorem in "Absolute integral closure in positive characteristic" Adv. Math. 210 (2007) by C. Huneke and G. Lyubeznik,and as a application,we construct a new big Cohen-Macauley algebra.

1 Main theorem

1.1 theorem

Let R be a domain of prime characteristic.

(1) Let $\mathfrak{a}$ be an ideal of R and $[\eta]$ an element of $H^i_{\mathfrak{a}}(R)_{\mathrm{nil}}$. Then there exists a finitely generated generically Galois extension S of R, with $\mathrm{Gal}(S/R)$ a solvable group, such that $[\eta]$ maps to zero under the induced map $H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}}(S)$.

(2) Suppose $(R, \mathfrak{m})$ is a homomorphic image of a Gorenstein ring. Then there exists a finitely generated generically Galois extension S of R such that the induced map $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(S)$ is zero for each $i < \dim R$.

Here we say *"S is a generically Galois extension of R"* if S is an extension domain that is integral over R and the extension of fractions fields is Galois. In this case, $\mathrm{Gal}(S/R)$ will denote the Galois group of the corresponding extension of fraction fields.And $H^i_{\mathfrak{a}}(R)_{\mathrm{nil}}$ means the union of $\ker F^e$:$H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}}(R)$ for all positive integer e.

1.2 remark

(2) shows the condition is close to characteristic 0 ,but the statement of this type is never valid in characteristic 0.
For example,take R normal ring in characteristic 0,then for every finite extension S,the inclusion map splits as R-module by the trace map. Therefore the maps on local cohomology also split,especially is injection.

2 History and corollary

C. Huneke and G. Lyubeznik proved a weaker version of (2) above.
They proved existence of S which is just finite over R to prove the existence of big Cohen-Macauley algebra.
Big Cohen-Macauley algebra is a R-algebra S(not necessarily Noether) such that every system of parameter of R is regular sequence on S.
They proved the integral closure in a algebraic closure of the quotient field of R(denoted R^+) is big Cohen-Macauley algebra.With the same discussion,we can prove the existence of another big Cohen-Macauley algebra.

2.1 Existence of another Big Cohen-Macauley algebra

Let $(R, \mathfrak{m})$ is a homomorphic image of a Gorenstein ring.
and R^{+sep} be the separable closure in a algebraic closure of the quotient field of R.
Then $R^{+sep}$is big Cohen-Macauley algebra.

3 Application of existence of big Cohen-Macauley algebra

Big Cohen-Macauley algebra is first introduced by M. Hochster and C. Huneke in "Infinite integral extensions and big Cohen-Macaulay algebras", Ann. of Math(1992).The existence is known in the case of positive characteristic,equicharacteristic or the dimension is lower than 4.It is known that the existence of big Cohen-Macauley algebra implies various conjectures below.

3.1 Direct summand conjecture

Let R be a regular Noetherian ring and S be a module-finite R-algebra,then R is a direct summand of S as an R-module.

3.2 Monomial conjecture

Let S be a local ring and $x_1, \ldots, x_n$ be a system of parameters for S,then for every integer k, $(x_1 \cdots x_n) \notin (x_1^{k+1}, \ldots, x_n^{k+1})S$.

3.3 Cohen-Macauleyness of direct summands of regular rings

Let R be a Noetherian ring and S be a R-algebra.Assume R is a direct summand of S as an R-module, then R is Cohen-Macauley.

4 Graded case

Let R^{+GR} be the subalgebra of R^+ which is generated by homogeneous integral element over R. So R^{+GR} is $\mathbb{Q}$ -graded.Hochster and Huneke proved this algebra is big Cohen-Macauley algebra. Since we proved R^{+sep} is big Cohen-Macauley,it is natural to ask whether $R^{+GR,sep} = R^{+GR} \cap R^{+sep}$ is big Cohen-Macauley algebra.

4.1 Examples

Let R be the Rees ring

$$\frac{\overline{\mathbb{F}}_2[x,y,z]}{(x^3+y^3+z^3)}[xt, yt, zt]$$

with the $\mathbb{N}$-grading where the generators x, y, z, xt, yt, zt have degree 1.

(1) the ring $R^{+GR,sep}$is not big Cohen-Macaulay R-algebra.

(2) let S be a graded Cohen-Macaulay ring with $R \subseteq S \subseteq R^{+\mathrm{GR}}$. We prove that S is not finitely generated over R.

(2) means nonexistence of small(Noether) Cohen-Macauley algebra in graded case,which is open in general case.
Huneke and Lyubeznik calculated only lower cohomology,but we calculate the top cohomology.

4.2 Proposition on top local cohomology

Let R be an $\mathbb{N}$-graded domain that is finitely generated over a field R_0 of positive characteristic. Set $d = \dim R$. Then the submodule $[H^d_{\mathfrak{m}}(R)]_{\geq 0}$ maps to zero under the induced map

$$H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(R^{+\mathrm{GR}}).$$

Hence, there exists a finitely generated graded extension S of R, contained in $R^{+\mathrm{GR}}$, such that $[H^d_{\mathfrak{m}}(R)]_{\geq 0} \longrightarrow H^d_{\mathfrak{m}}(S)$ is zero.

When R is a section ring of a projective variety,we can understand this proposition more geometorically by using the correspondence between sheaf cohomology and local cohomology.

4.3 Geometrical application

Let X denote a projective variety over a field K,then there is a finite surjective morphism of projective varieties $f : Y \to X$, such that the induced map $H^d(X, \mathcal{O}_X(t)) \to H^d(X, f^*\mathcal{O}_X(t))$ is zero map for all nonnegative integer t.

5 Thanks for reading this poster and feel free to ask questions.